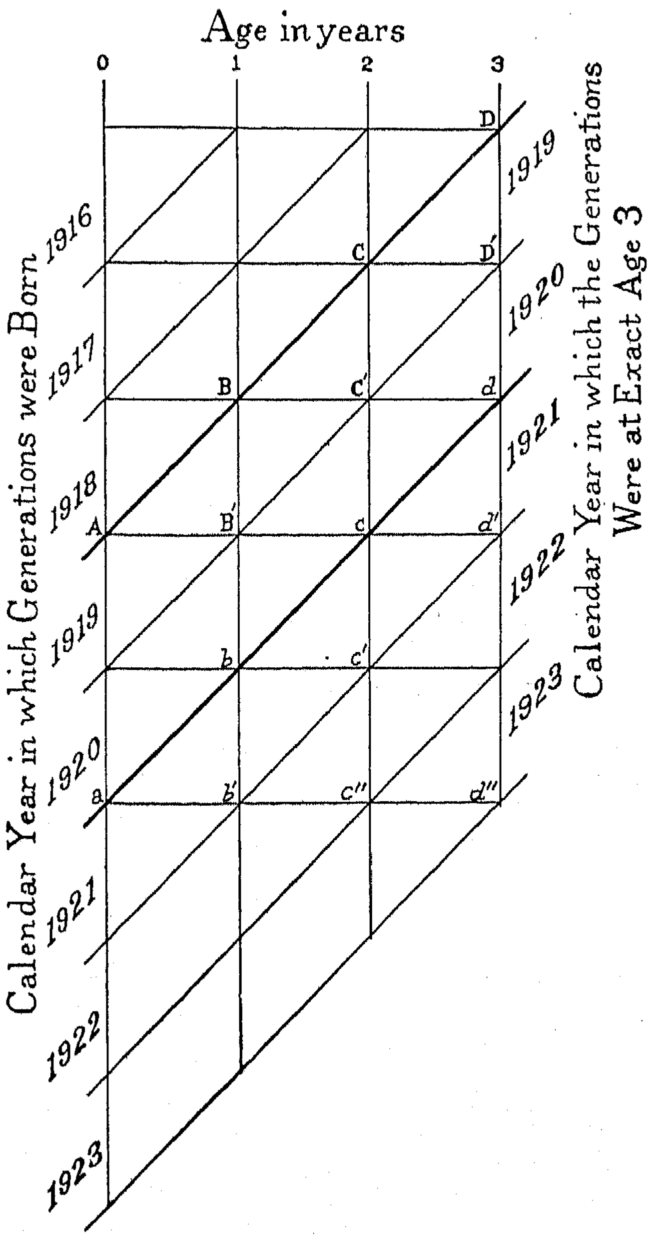
Part II.—METHODS USED AND ACTUAL COMPUTATION.

A.—EXPLANATION OF METHODS USED.

THEORY OF METHOD USED IN OBTAINING RATES OF MORTALITY AT AGES UNDER 3 YEARS.

7. Diagrams 1 to 3 represent the progress of generations. These diagrams are fully explained in sections 96, 106, and 109, pages 329, 338, and 340 of the United States Life Tables, 1890, 1901, 1910, and

Diagram 1.—Movement of Generations Represented Graphically.



1901–1910. In brief, the ages of the generations are measured by vertical lines along the horizontal axis. In the diagram, time in calendar years is measured by the diagonal lines which are at right angles to the bisector of the angle between the vertical and horizontal axes. This bisector is not drawn in these diagrams. Thus the generations begin along the vertical axis at age 0 and move horizontally to the right. See Diagrams 2 and 3, pages 33 and 37. In any generation many die under 1 year of age; for instance, of those born in 1916, E_0^{1916}, some die in 1916, lD_0^{1916}, and some in 1917, eD_0^{1917}. Of those who survive to exact age 1 year, E_1^{1916}, many die between exact ages 1 and 2 years, some in 1917, lD_1^{1917}, and some in 1918, eD_1^{1918}. Likewise, the deaths among the survivors to exact age 2 years, E_2^{1916}, occur in 1918, lD_2^{1918}, and some in 1919, eD_2^{1919}.

If a census be taken of these generations at any time, for instance, January 1, 1919, the children under 3 years of age enumerated would be those who were born between January 1, 1916, and January 1, 1919, who had not died before January 1, 1919. Thus the children between 2 and 3 years of age on January 1, 1919, would be that part of the 1916 generation, E_0^{1916}, which was not included in $lD_0^{1916}+eD_0^{1917}+lD_1^{1917}+eD_1^{1918}+lD_2^{1918}$.

The method used to derive the formula for the annual rate of mortality at each year of age under 3 is a modification of the method suggested by Mr. Robert Henderson. The rate of mortality of the generation that attains age x during the calendar period is by definition $q_x=d_x/l_x$, where l_x is the number that attain age x during the calendar period and d_x is the number of deaths that occur among the l_x persons before they become aged exactly $x+1$ years. Part of these d_x occur in the year following the calendar period of years. An illustration of this is afforded in Diagram 2. Thus, $E_0^{1919}+E_0^{1920}$, or E_0, is the number of children born during the calendar period 1919–1920, or the number that attain age 0 during that period. Before this generation has become aged exactly 1 year, d_0 of them have died, lD_0^{1919} in 1919, $eD_0^{1920}+lD_0^{1920}$ in 1920, and eD_0^{1921} in 1921. On the other hand, some of the deaths under 1 year of age in 1919–1920, eD_0^{1919}, were of children born in 1918. Accordingly, it appears that the number of deaths under 1 year of age during 1919 and 1920 is

$$D_0=eD_0^{1919}+lD_0^{1919}+eD_0^{1920}+lD_0^{1920}$$

and that in the generation born in 1919–1920 before it attains exact age 1 year is

$$d_0=lD_0^{1919}+eD_0^{1920}+lD_0^{1920}+eD_0^{1921}.$$

Thus the difference between the number of deaths under 1 year of age in the calendar period 1919–1920 and in the generation born in that period is

$$D_0-d_0=eD_0^{1919}-eD_0^{1921}=r_0^{1919}P_{1919}^{0/1}-r_0^{1921}P_{1921}^{0/1},$$

where r_0^y is the ratio of the number of deaths under 1 year of age in the calendar year y among those born in the previous year, $y-1$, to $P_y^{0/1}$.

From Diagram 2 it appears that the deaths under 1 year of age in the calendar period 1919–1920 must occur among the $P^{0/1}_{1919}+E^{1919}_0+E^{1920}_0$ children and that the $P^{0/1}_{1919}$ and $P^{0/1}_{1921}$ children lived only a part of their lives between birth and 1 year of age in the period 1919–1920. Hence the rate of mortality under 1 year of age in the *calendar period* 1919–1920 must be

$$q^c_0=D_0/E'_0,$$

where E'_0 may be called the equivalent generation which corresponds to the deaths D_0.

In the special case where the force of mortality at each age in triangle AB′B, Diagram 1, is equal to that at the corresponding age in triangle *ab′b* and in quadrilateral AabB′, the rates of mortality under 1 year of age in 1919–1920 and in the generation born in 1919–1920 would be the same, and r^{1919}_0, r^{1920}_0, r^{1921}_0 would all be equal.

Then the equation

$$D_0-d_0=r^{1919}_0P^{0/1}_{1919}-r^{1921}_0P^{0/1}_{1921}$$

may be written

$$D_0=d_0+r^{1919}_0\delta_0, \text{ where } \delta_0 \text{ is } P^{0/1}_{1919}-P^{0/1}_{1921},$$

so that

$$E'_0q^c_0=E_0q_0+r^{1919}_0\delta_0.$$

Then since $q_0=q^c_0$,

$$E'_0=E_0+k_0\delta_0, \text{ where } k_0 \text{ is } r^{1919}_0/q_0.$$

When k_0 equals $\frac{1}{2}$, this formula for the approximate value of the equivalent generation is that given in equation (22) of the United States Life Tables, 1890, 1901, 1910, 1901–1910, page 337.

By reasoning similar to the above approximate values for rate of mortality between exact ages 1 and 2 years and between 2 and 3 years in 1919–1920 are shown to be, respectively,

$$q^c_1=D_1/(E_1+r_1\delta_1/q_1) \text{ and } q^c_2=D_2/(E_2+r_2\delta_2/q_2),$$

where E_1 and E_2 are the numbers of children that attain ages 1 and 2 years, respectively, in the calendar period 1919–1920.

Where the rate of mortality does not change very rapidly between ages x and $x+1$, r_x/q_x is very nearly equal $\frac{1}{2}$. However, the rate of mortality under 1 year of age does change very rapidly, and for this reason k_0 was determined from infant mortality statistics given in Table 13 of Birth Statistics of the Birth Registration Area of the United States in each year from 1918 to 1921, published by the Bureau of the Census. The statistics from which the value for k_0 was determined were from the same area as that covered by the 1919–1920 life tables, except Rhode Island, Illinois, Missouri, Tennessee, and Hawaii, and should, therefore, be a very good average for these tables. The results obtained were 0.275 for males and 0.280 for females. While the rate of mortality under 1 year of age has been very much lowered between 1909 and 1919, that under 1 day of age has not changed much. The consequence is that the per cent of born and died in a calendar year has been raised, so that k_0 has changed from about $33\frac{1}{3}$ per cent in 1909–1911 to about 28 per cent in 1919–1920.[1]

Unfortunately no statistics are available to determine k_1 and k_2. However, there is no evidence of irregularity in the lowering of the rates of mortality during the age periods 1 to 2 years and 2 to 3 years, and so k_1 and k_2 were set equal to $\frac{1}{2}$, the ratio used for the 1909–1911 life tables. See United States Life Tables, 1890, 1901, 1910, and 1901–1910, page 343, equations (30).

From Diagram 2 it will be seen that

$$E_1=E_0+P^{0/1}_{1919}-P^{0/1}_{1921}-D_0=E_0+\delta_0-D_0,$$

while

$$E_2=E_1+P^{1/2}_{1919}-P^{1/2}_{1921}-D_1=E_1+\delta_1-D_1.$$

For the convenience of the operator the three equations just derived were expanded. Let G_x represent the denominator in the equation $q_x=D_x/(E_x+k_x\delta_x)$. Then

$$\begin{aligned}
G_0&=E_0+k_0\delta_0,\\
G_1&=E_1+\tfrac{1}{2}\delta_1=E_0+\delta_0+\tfrac{1}{2}\delta_1-D_0\\
&=G_0+(1-k_0)\delta_0-D_0+\tfrac{1}{2}\delta_1.\\
G_2&=E_2+\tfrac{1}{2}\delta_2=E_1+\delta_1+\tfrac{1}{2}\delta_2-D_1\\
&=G_1-D_1+\tfrac{1}{2}(\delta_1+\delta_2).
\end{aligned}$$

Therefore, the three equations become

$$q_0=D_0/G_0, \qquad (1)$$

$$q_1=D_1/[G_0+(1-k_0)\delta_0-D_0+\tfrac{1}{2}\delta_1], \qquad (2)$$

$$q_2=D_2/[G_1-D_1+\tfrac{1}{2}(\delta_1+\delta_2)]. \qquad (3)$$

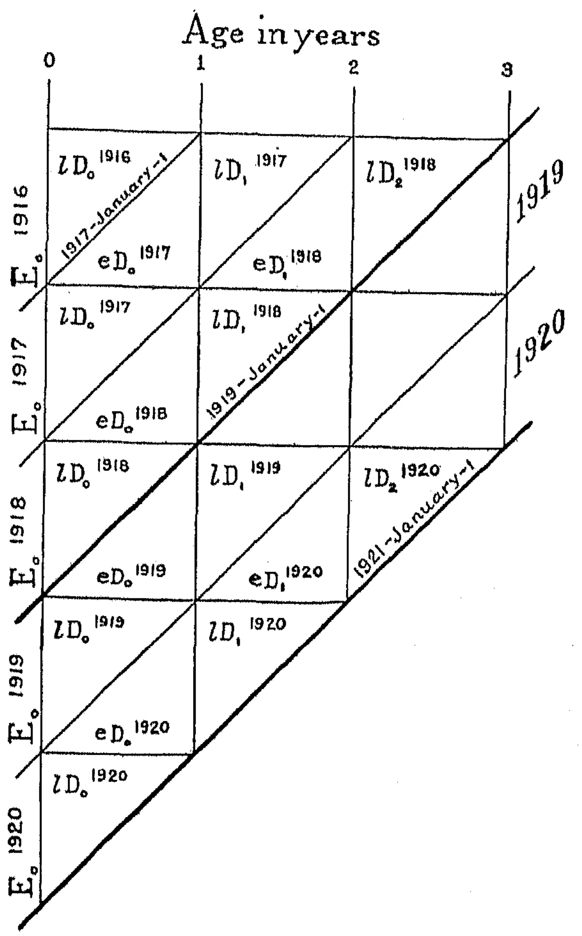
METHOD USED TO DETERMINE DIFFERENCE BETWEEN POPULATION IN SAME AGE INTERVAL AT BEGINNING AND END OF PERIOD.

8. Only the *differences* between the populations at corresponding ages on January 1, 1919, and on January 1, 1921, were used. Therefore, populations derived from birth and death statistics are sufficient since the effect of migration on the number of children under 1 year of age on January 1, 1919, should be about the same as that on the number of children under 1 year of age on January 1, 1921, and this effect would be cancelled out in a difference. The same is true of children between 1 and 2 years of age on January 1, 1919, and January 1, 1921, and also of children between 2 and 3 years on those dates. The method of determining these populations from birth and death statistics is based on the method used to determine the number of births for the United States Life Tables, 1890, 1901, 1910, 1901–1910,

[1] Mr. Henderson bases the ratio of the number of deaths under 1 year of age in the calendar year y among those born in the previous year, $y-1$, upon the statistics for two consecutive calendar years, so that he sets $r^{1920\text{-}21}_0=r^{1919\text{-}20}_0=r_0=(eD^{1919}_0+eD^{1920}_0)/(P^{0/1}_{1919}+P^{0/1}_{1920})$. The value for k_0 derived from this value of r_0 is 0.288 for males and 0.290 for females. While as a rule the value of k_0 seems to be decreasing with time, it probably varies considerably from locality to locality and from race to race. However, no statistics were available for the separate localities and races from which their values of k_0 could be determined.

explained in section 109, page 340. Instead of adding populations to deaths to find the number of births, deaths were subtracted from the births to obtain populations. E_0^y in Diagram 2 represents the number of births in any calendar year y; lD_x^y, the number of deaths between ages x and $x+1$ in that year of those who were born in the *later* calendar year, and eD_x^y, the number of deaths between ages x and $x+1$ in that year of those who were born in the *earlier* calendar year.

DIAGRAM 2. GRAPHIC REPRESENTATION OF RELATION BETWEEN BIRTH AND DEATH RECORDS AND CENSUS STATISTICS FOR 1919–1920 LIFE TABLES.



From this it appears that the population under 1 year of age on January 1, 1919, is $P_{1919}^{0/1} = E_0^{1918} - lD_0^{1918}$ and the population under 1 year of age on January 1, 1921, is $P_{1921}^{0/1} = E_0^{1920} - lD_0^{1920}$.

As in equations (1) to (3) on page 32, the expression $(P_{1919}^{x/x+1} - P_{1921}^{x/x+1})$, is designated by δ_x. Consequently,

$$\delta_0 = (-E_0^{1920} + E_0^{1918}) + (-lD_0^{1918} + lD_0^{1920}). \qquad (4)$$

The population between 1 and 2 years of age on January 1, 1919, is

$$P_{1919}^{1/2} = E_0^{1917} - lD_0^{1917} - eD_0^{1918} - lD_1^{1918}$$

and the population between 1 and 2 years of age on January 1, 1921, is

$$P_{1921}^{1/2} = E_0^{1919} - lD_0^{1919} - eD_0^{1920} - lD_1^{1920}.$$

Hence,

$$\delta_1 = (-E_0^{1919} + E_0^{1917}) + (-lD_0^{1917} + lD_0^{1919}) + (-eD_0^{1918} + eD_0^{1920}) + (-lD_1^{1918} + lD_1^{1920}). \qquad (5)$$

The population between 2 and 3 years of age on January 1, 1919, is

$$P_{1919}^{2/3} = E_0^{1916} - lD_0^{1916} - eD_0^{1917} - lD_1^{1917} - eD_1^{1918} - lD_2^{1918}$$

and the population between 2 and 3 years of age on January 1, 1921, is

$$P_{1921}^{2/3} = E_0^{1918} - lD_0^{1918} - eD_0^{1919} - lD_1^{1919} - eD_1^{1920} - lD_2^{1920}.$$

Accordingly,

$$\delta_2 = (-E_0^{1918} + E_0^{1916}) + (-lD_0^{1916} + lD_0^{1918}) + (-eD_0^{1917} + eD_0^{1919}) + (-lD_1^{1917} + lD_1^{1919}) + (-eD_1^{1918} + eD_1^{1920}) + (-lD_2^{1918} + lD_2^{1920}) \qquad (6)$$

Then each number of deaths in Table 17, pages 58 to 61 was divided into lD and eD by applying the percentages given in the United States Life Tables, 1890, 1901, 1910, 1901–1910, page 340, Table 109, and the resulting lD and eD were entered in different colored ink just below the D from which they were derived. The method of taking these values of lD, eD, and E_0^y from the table in computing infant mortality is illustrated in tape 16, page 39.

METHOD USED TO OBTAIN RATES OF MORTALITY FOR AGES BETWEEN ADOLESCENCE AND OLD AGE.

9. In obtaining graduated rates of mortality for each fifth year of age from 12 to 92, the formula used was that employed by Mr. George King[1] for finding the graduated central value of a fifteen term series. Equations (82) in the United States Life Tables, 1890, 1901, 1910, 1901–1910, page 390, section 180, were transformed for the convenience of operators as follows:

	$-\Delta T_{x-7}$	$-\Delta T_{x-2}$	$-\Delta T_{x+2}$
$-200\Delta T_{x-2}$ =		-200	
$-(-8\Delta^3 T_{x-7})$ =	-8	$+16$	-8

Since $-\Delta T_x$ is the sum of the population aged x to $x+4$ on January 1, 1920, the symbol $P_{1920}^{x/x+4}$ is used, and

$$10^3 L_{x+2} = (-10+2)\ (P_{1920}^{x-5/x-1} - 2P_{1920}^{x/x+4} + P_{1920}^{x+5/x+9}) + 200P_{1920}^{x/x+4}. \qquad (7)$$

[1] Supplement to the Seventy-fifth Annual Report of the Registrar-General of Births, Deaths, and Marriages in England and Wales, Part I—Life Tables, page 49, section 2.

Also since $-\Delta(2l)_x$ is the sum of the deaths occurring between ages x and $x+5$ during the two calendar years 1919 and 1920, the symbol $D^{1919\text{-}20}_{x/x+4}$ is used, and

$$10^3(2d)_{x+2}=(-10+2)\,(D^{1919\text{-}20}_{x-5/x-1}-2D^{1919\text{-}20}_{x/x+4}+D^{1919\text{-}20}_{x+5/x+9})$$
$$+200D^{1919\text{-}20}_{x/x+4}. \qquad (8)$$

No knowledge of differencing, negative values, or fractions is required to use the equations in this form. The method of using them is illustrated on page 39, tapes 18 and 19.

METHOD USED TO JOIN MORTALITY OF CHILDREN UNDER 3 YEARS OF AGE WITH THAT IN THE MAIN TABLE.

10. The formula discussed in section 9 is for finding the central or eighth term of a fairly symmetrical series of fifteen values. The derivation of a formula for interpolating the third term in this series of fifteen values is similar to that for interpolating the eighth term of the series. If u_2 be the third term in a series of fifteen terms, u_0, u_1, and so on up to u_{14}, and $y_n=\sum_{x=n}^{x=14}u_x$, so that $\Delta y_n=-\sum_{x=n}^{x=n+4}u_x$, then

$$-u_2=y_3-y_2$$
$$y_3=y_0+\tfrac{3}{5}\Delta y_0-\tfrac{3}{25}\Delta^2 y_0+\tfrac{7}{125}\Delta^3 y_0$$
$$y_2=y_0+\tfrac{2}{5}\Delta y_0-\tfrac{3}{25}\Delta^2 y+\tfrac{8}{125}{}^3 y$$
$$-u_2=\tfrac{1}{5}\Delta y_0-\tfrac{1}{125}\Delta^3 y_0$$
$$=.2\Delta y_0-.008\Delta^3 y_0$$

or

$$-10^3u_2=200\sum_{x=0}^{x=4}u_x-8\left(\sum_{x=0}^{x=4}u_x-2\sum_{x=5}^{x=9}u_x+\sum_{x=10}^{x=14}u_x\right).$$

When L_7 and $(3d)_7$ are substituted for u_2, and $P^{x/x+4}_{1920}$ and $D^{1919\text{-}20}_{x/x+4}$ are substituted for Σu_x, and age 5 is taken as 0, the following two equations are obtained:

$$10^3L_7=200P^{5/9}_{1920}+(-10+2)(P^{5/9}_{1920}-2P^{10/14}_{1920}+P^{15/19}_{1920}) \qquad (9)$$
$$10^3(3d)_7=200D^{1919\text{-}20}_{5/9}+(-10+2)(D^{1919\text{-}20}_{5/9}-2D^{1919\text{-}20}_{10/14}$$
$$+D^{1919\text{-}20}_{15/19}) \qquad (10)$$

These formulas were used to determine graduated populations and deaths at age 7, and the results were found to be fairly good and served to join life table values of children under 3 years of age with those beginning at age 12. See values in Table 2, page 10.

METHOD USED TO EXTEND THE PROBABILITIES OF LIVING TO EXTREME OLD AGE.

11. The plan suggested by Mr. George King[1] was followed for the most part, in some cases a constant third difference being used when the fourth differences did not seem suitable. The logarithms of the last seven probabilities of living, given at quinquennial ages, were differenced four times and the largest negative fourth difference or the last negative fourth difference was used to extend these probabilities of living over periods of five years up to age 112. The processes used are illustrated in tapes 24 to 34, pages 43 and 45.

METHOD USED TO DERIVE $\log {}_5p_x$ FROM $\log p_x$ AT EVERY FIFTH YEAR OF AGE AND DETERMINATION OF l_x COLUMN.

12. The formulas used for this process are those given by Mr. George King,[1] but the equations were put in another form that requires no differencing and is better suited for machine work. For convenience and reference equations (i) and (iii) are copied here.

$$w_5=5u_0+7\Delta u_0+1.6\Delta^2u_0-.2\Delta^3u_0 \qquad \text{(i)}$$
$$w_0=5u_0+2\Delta u_0-0.4\Delta^2u_0+.2\Delta^3u_0, \qquad \text{(iii)}$$

where $w_5=\sum_{x=5}^{x=9}u_x$ and $w_0=\sum_{x=0}^{x=4}u_x$. These two equations were transformed by substituting for the leading differences of u_0 their equivalents in terms of the quinquennial values of u_x. This work is indicated below.

Transformation of equation (iii)

	u_0	u_5	u_{10}	u_{15}
$5.0u_0=$	$+5.0$			
$2.0\Delta u_0=$	-2.0	$+2.0$		
$-0.4\Delta^2u_0=$	-0.4	$+0.8$	-0.4	
$0.2\Delta^3u_0=$	-0.2	$+0.6$	-0.6	$+0.2$
Total, $w_0=+2.4u_0+3.4u_5-1.0u_{10}+0.2u_{15}$				

or

$$10w_0=24u_0+34u_5-10u_{10}+2u_{15}$$
$$=24(u_0+u_5)+10u_5-10u_{10}+2u_{15} \qquad (11)$$

Transformation of equation (i)

	u_0	u_5	u_{10}	u_{15}
$5.0u_0=$	$+5.0$			
$7.0\Delta u_0=$	-7.0	$+7.0$		
$1.6\Delta^2u_0=$	$+1.6$	-3.2	$+1.6$	
$-0.2\Delta^3u_0=$	$+0.2$	-0.6	$+0.6$	-0.2
Total, $w_5=-0.2u_0+3.2u_5+2.2u_{10}-0.2u_{15}$				

or

$$10w_5=-2u_0+32u_5+22u_{10}-2u_{15}$$
$$=2[-u_0+11(u_5+u_{10})-u_{15}]+10u_5 \qquad (12)$$

Section 36, page 44, shows that the computations indicated in equations (11) and (12) may be readily performed upon an adding machine.

Mr. Robert Henderson suggested that the curve of probabilities of living between ages 2 and 7 is so skew that formula (iii) should be adjusted by determining the coefficient of Δ^3u_0 from known values of $\log {}_5p_2$.

[1] Supplement to the Seventy-fifth Annual Report of the Registrar-General of Births, Deaths, and Marriages in England and Wales, Part I—Life Tables, pages 26 to 28.

The values for the coefficient of $\Delta^3 u_0$, computed from a number of the United States 1910 life tables, varied slightly about unity. Values for the coefficient $\Delta^3 u_0$, computed in the same way from known values of $\log {}_5p_7$ in these same life tables, all varied only slightly from 0.2. Accordingly, $\log {}_5p_7$ was determined by using equation (11) and $\log {}_5p_2$ by using equation (11a), which is derived from a modification of equation (iii)—that is, from

$$w_0 = 5u_0 + 2\Delta u_0 - 0.4\Delta^2 u_0 + \Delta^3 u_0. \qquad \text{(iiia)}$$

Transformation of equation (iiia).

	u_0	u_5	u_{10}	u_{15}
$5u_0 =$	$+5.0$			
$2\Delta u_0 =$	-2.0	$+2.0$		
$-0.4\Delta^2 u_0 =$	-0.4	$+0.8$	-0.4	
$\Delta^3 u_0 =$	-1.0	$+3.0$	-3.0	$+1.0$
Total, $w_0 =$	$+1.6u_0$	$+5.8u_5$	$-3.4u_{10}$	$+1.0u_{15}$

or

$$10w_0 = +17\ (u_0 + 4u_5 - 2u_{10}) - u_0 - 10u_5 + 10u_{15}$$
$$= (20\text{-}3)(u_0 + 4u_5 - 2u_{10}) - u_0 - 10u_5 + 10u_{15}. \qquad \text{(11a)}$$

When $\log {}_5p$ is substituted for w and $\log p$ for u in equations (11a), (11), and (12), they become

$$10\log {}_5p_2 = (20\text{-}3)(\log p_2 + 4\log p_7 - 2\log p_{12}) - \log p_2 - 10\log p_7 + 10\log p_{17} \qquad (13)$$

$$10\log {}_5p_7 = 24(\log p_7 + \log p_{12}) + 10\log p_{12} - 10\log p_{17} + 2\log p_{22} \qquad (14)$$

$$10\log {}_5p_{12} = 2[-\log p_7 + 11(\log p_{12} + \log p_{17}) - \log p_{22}] + 10\log p_{12} \qquad (15)$$

$$10\log {}_5p_{17} = 2[-\log p_{12} + 11(\log p_{17} + \log p_{22}) - \log p_{27}] + 10\log p_{17} \qquad (16)$$

and so on.

100,000 was taken as the radix of the table, and to 5, its logarithm, $\log p_0$, $\log p_1$, $\log {}_5p_2$, $\log {}_5p_7$, and so on, were added, subtotals being taken after each addition. These subtotals are the logarithm of l_x.

METHOD OF DETERMINING EXPECTATION OF LIFE FROM SURVIVORS AT EVERY FIFTH YEAR OF AGE.

13. Equations (11) and (12) were transformed by substituting $N'_{\overline{5}|}$ for w and l for u, and the following equations were obtained:

$$10N'_{2:\overline{5}|} = 24(l_2 + l_7) + 10l_7 - 10l_{12} + 2l_{17} \qquad (17)$$
$$10N'_{7:\overline{5}|} = 2[-l_2 + 11(l_7 + l_{12}) - l_{17}] + 10l_7 \qquad (18)$$
$$10N'_{12:\overline{5}|} = 2[-l_7 + 11(l_{12} + l_{17}) - l_{22}] + 10l_{12} \qquad (19)$$

and so on to

$$10N'_{(w-10):\overline{5}|} = 2[-l_{w-15} + 11\ (l_{w-10} + l_{w-5}) - l_w] + 10l_{w-10}. \qquad (20)$$

w designates the age of the last l_x, determined by the method described above, which had a value as large as 0.5. Any value between 0.5 and 1.0 was taken as 1.0. It will be noted that $N'_{(w-5):\overline{5}|}$ and $N'_{w:\overline{5}|}$ can not be determined by this formula. The general rule for obtaining $N'_{w-5:\overline{5}|}$ was to use 0 for l_{w+5}, thus forming the equation

$$10N'_{(w-5):\overline{5}|} = 2[-l_{w-10} + 11(l_{w-5} + l_w) - 0] + 10l_{w-5}. \qquad (21)$$

Sometimes, however, a negative value was obtained by using this formula and in that case $N'_{(w-5):\overline{5}|}$ was determined as follows: $\log p_{w-5}$ was added four times to $\log l_{w-5}$, a subtotal being taken after each addition and a total at the end. These three subtotals and the total are the logarithms of the approximate values of

$$l_{w-4},\ l_{w-3},\ l_{w-2},\ l_{w-1}. \quad \text{Then } N'_{(w-5):\overline{5}|} = \sum_{x=w-5}^{x=w-1} l_x. \qquad \text{(21a)}$$

It was never necessary to use (21a) for $N'_{(w-5):\overline{5}|}$ except when $l_w = 1$. In that case $N'_{w:\overline{5}|}$ was simply taken as 1. When l_w was greater than 1, $N'_{w:\overline{5}|}$ was determined according to the process outlined for (21a). That is, $\log p_w$ was added four times to $\log l_w$, a subtotal being taken after each addition with a total at the end. Whenever any of these subtotals became less than 999|698980000, which is log 0.5, the additions were stopped, since all values of l_x lower than 0.5 were taken as 0. Since $l_w = 1$ in tape 39, page 49, $N'_{w:\overline{5}|}$ is taken as 1, and the process indicated by (21a) was not needed.

Then to obtain N'_x, these values of $N'_{x:\overline{5}|}$ were summed, beginning with $N'_{w:\overline{5}|}$, and a subtotal was taken after each addition with a total at the end. The equation for the complete expectation of life is then

$$\mathring{e}_x = N'_x / l_x - 0.5. \qquad (22)$$

B.—ACTUAL COMPUTATION OF ABRIDGED LIFE TABLE FOR MALES IN THE STATE OF NEW YORK: 1909–1911.

PRELIMINARY STATEMENT.

14. To illustrate the process of constructing these abridged life tables, photographs of the actual computation of the New York Male, 1909–1911, Life Table, are shown on pages 39 to 52. The work of compiling the original statistics and that indicated in Table 12 is not given, but no other part of these computations is omitted except the divisions performed on computing machines to obtain the 21 rates of mortality in tapes 17 and 22 and the 22 expectations of life in tape 43, the multiplication of δ_0 by k_0, and also the work of looking up the antilogarithms in tape 37. The computations are on 28 tapes, each tape being described in a section having same number as tape. Ages and complete headings were copied on many of the tapes which are not needed in actual computations.

Checks for the comparer are designated by numbers enclosed in circles. Thus the 1 and 2 opposite the totals in tapes 21 and 22, respectively, and also opposite the totals in tape 23 indicate that the numbers marked by the same symbol should agree.

Throughout this description the word "complements" is used freely to mean any two numbers whose sum is any power of ten instead of only for those whose sum is unity. The use of these "complements" is a great aid to speed and accuracy, for no attention need be given to signs.

PREPARATION OF STATISTICS FOR DETERMINATION OF RATES OF MORTALITY OF CHILDREN UNDER 3 YEARS OF AGE.

15. The first step in the computation of rates of mortality at ages under 3 years was to arrange the births and deaths as in Table 12. The numbers of registered births were copied from state reports. The adjusted number of births for the period 1909–1911 was taken from the computations by the extended method. (See United States Life Tables, 1890, 1901, 1910, 1901–1910, page 373, tape 142.) The ratio between this adjusted number of births and the sum of the number of births registered was determined, 346, 664/327, 314 =1.059117545, and this was applied to the numbers of registered births in 1906, 1907, and 1908 to obtain the adjusted number of births for each of these years.

The number of deaths by single years of age under 3 during each of the calendar years, 1906 through 1911, was obtained from the Mortality Statistics for each of these years, published by the Bureau of the Census. 72 per cent of the deaths under age 1 year were assumed to be born in the *later* calendar year, lD_0, and 28 per cent in the *earlier* calendar year, eD_0; 59 per cent of the deaths in age interval 1–2 years were assumed to be born in the *later* calendar year, lD_1, and 41 per cent in the *earlier* year, eD_1; 53 per cent of the deaths in age interval 2–3 years were assumed to be born in the *later* calendar year, lD_2, and 47 per cent in the *earlier* year, eD_2. This is in accordance with the constants used in construction of United States Life Tables, 1890, 1901, 1910, 1901–1910, given in Table 109, page 340, of the volume of this title.

TABLE 12.—STATISTICS FROM WHICH RATES OF MORTALITY OF CHILDREN UNDER 3 YEARS OF AGE WERE DETERMINED FOR THE NEW YORK MALE LIFE TABLE, 1909–1911.

Calendar year	1906	1907	1908	1909	1910	1911
Number of births registered	93, 988	100, 522	104, 992	104, 382	109, 229	113, 703
Adjusted number of births	99, 544	106, 465	111, 199	111, 666	115, 948	119, 050
Number of deaths, 0–1, D_0	15, 209	15, 432	14, 632	14, 569	15, 234	14, 040
Born in later year, lD_0	10, 950	11, 111	10, 535	10, 490	10, 968	10, 109
Born in earlier year, eD_0	4, 259	4, 321	4, 097	4, 079	4, 266	3, 931
Number of deaths, 1–2, D_1	....	3, 414	3, 229	3, 523	3, 401	2, 993
Born in later year, lD_1	....	2, 014	1, 905	2, 079	2, 007	1, 766
Born in earlier year, eD_1	....	1, 400	1, 324	1, 444	1, 394	1, 227
Number of deaths, 2–3, D_2	....	....	1, 442	1, 484	1, 545	1, 320
Born in later year, lD_2	....	....	764	....	....	700
Born in earlier year, eD_2	....	....	678	....	....	620

DIFFERENCES BETWEEN POPULATIONS AT CORRESPONDING AGES ON JANUARY 1, 1909, AND JANUARY 1, 1912.

16. It was necessary to determine first the difference between the populations at corresponding ages on January 1, 1909, and January 1, 1912. Formulas for this work, (4), (5), and (6), were derived on page 33. The New York Male, 1910, Life Table, is based on a three-year period, 1909–1911. Hence, to use these equations for the computations of this table, 1906 was substituted for 1916, 1907 for 1917, 1908 for 1918; then 1909 for 1918, 1910 for 1919, 1911 for 1920, and 1912 for 1921. (See Diagram 3.)

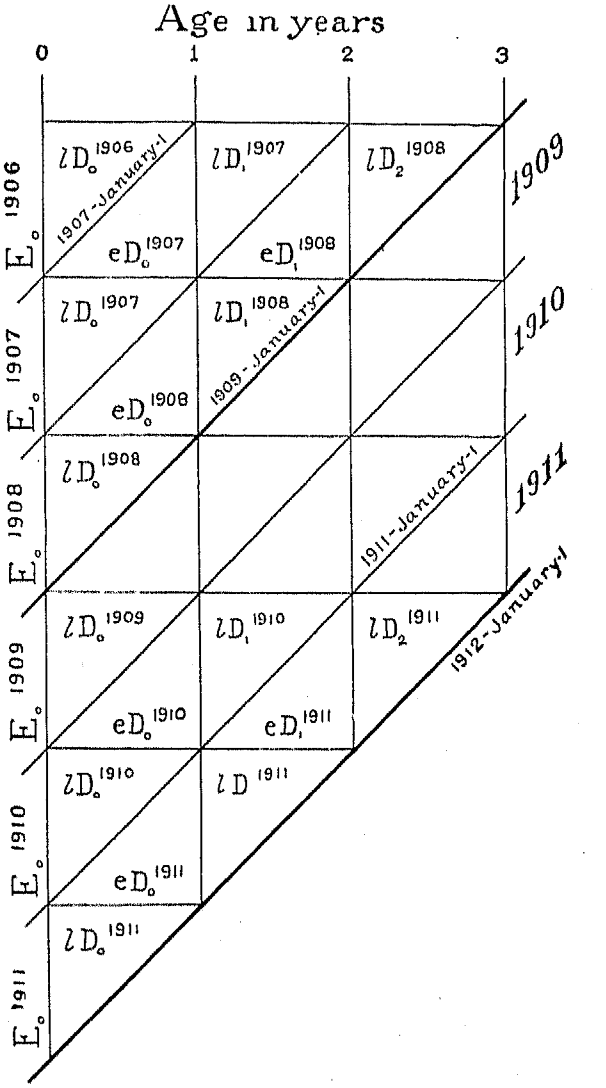
$$\delta_0 = (-E_0^{1911} + E_0^{1908}) + (-lD_{0/1}^{1908} + lD_{0/1}^{1911}) \tag{4a}$$

$$\delta_1 = (-E_0^{1910} + E_0^{1907}) + (-lD_{0/1}^{1907} + lD_{0/1}^{1910}) + (-eD_{0/1}^{1908} + eD_{0/1}^{1911}) + (-lD_{1/2}^{1908} + lD_{1/2}^{1911}) \tag{5a}$$

$$\delta_2 = (-E_0^{1909} + E_0^{1906}) + (-lD_{0/1}^{1906} + lD_{0/1}^{1909}) + (-eD_{0/1}^{1907} + eD_{0/1}^{1910}) + (-lD_{1/2}^{1907} + lD_{1/2}^{1910}) + (-eD_{1/2}^{1908} + eD_{1/2}^{1911}) + (-lD_{2/3}^{1908} + lD_{2/3}^{1911}). \tag{6a}$$

As will be noticed these equations are rather symmetrical and their values can be selected from Table 12 according to rule. The last group on the right is always $-lD_{x/x+1}^{1908} + lD_{x/x+1}^{1911}$, x being 0, 1, and 2. The next to the last group of deaths is always $-eD_{x/x+1}^{1908} + eD_{x/x+1}^{1911}$, x being 0 and 1; the second from the last group of deaths is always $-lD_{x/x+1}^{1907} + lD_{x/x+1}^{1910}$, x being 0 and 1; the third from the last group of deaths is $-eD_{x/x+1}^{1907} + eD_{x/x+1}^{1910}$; the fourth from the last group of deaths is $-lD_{x/x+1}^{1906} + lD_{x/x+1}^{1909}$. The group of E's is always for the same calendar years as the group of deaths adjoining, only the signs are changed. The additions were begun with the last group in each equation. The adding machine was split between the banks 9 and 10, and the lD's and eD's were set up from Table 12 on the adding machine

DIAGRAM 3.—GRAPHIC REPRESENTATION OF RELATION BETWEEN BIRTH AND DEATH RECORDS AND CENSUS STATISTICS FOR 1909-1911 LIFE TABLES.



in the same order as they appear in the equations, while the E's were added in the reverse order because of the change of sign.

Diagram 4 contains three outlines of Table 12 to indicate how to obtain the values for equations (4a), (5a), and (6a).

In actual computations Table 12 was extended in a straight line as in Table 17, which form was convenient for the operator and also for those preparing the statistics for a number of tables at the same time. It will be noted that negative quantities were set up on the left side of the machine and positive on the right. Hence, when all the values on the right side of each equation were set up, a subtotal was taken and the complement of the sum on the left side was set up on

DIAGRAM 4.—OUTLINE SHOWING ORDER IN WHICH BIRTHS AND DEATHS IN TABLE 12 SHOULD BE ADDED TO OBTAIN VALUES FOR EQUATIONS (4a), (5a), AND (6a).

		1906	1907	1908	1909	1910	1911
Equation (4a)	Adjusted Births			3			4
	Deaths 0-1, D_0						
	lD_0			2			1
	eD_0						
Equation (5a)	Adjusted Births		7			8	
	Deaths 0-1, D_0						
	lD_0		6			5	
	eD_0			4			3
	Deaths 1-2, D_1						
	lD_1			2			1
	eD_1						
Equation (6a)	Adjusted Births	11			12		
	Deaths 0-1 D_0						
	lD_0	10			9		
	eD_0		8			7	
	Deaths 1-2 D_1						
	lD_1		6			5	
	eD_1			4			3
	Deaths 2-3 D_2						
	lD_2			2			1
	eD_2						

both sides of the machine and a total taken in the case of the additions for (4a) and (5a) and a subtotal after additions for (6a). The left side of the machine should be cleared if the correct complement is set up. The remainders on the right are δ_0, δ_1, and δ_2, respectively. δ_1 is then set up below δ_2 and a total taken. δ_0 is then multiplied by k_0, which in 1910 was about $\frac{1}{3}$, and the product entered in pencil just below δ_0, and the difference $(1-k_0)\delta_0$ is written just below the product $k_0\delta_0$. Then $\frac{1}{2}$ of δ_1 and also of $(\delta_1+\delta_2)$ is copied just below them.

DETERMINATION OF RATES OF MORTALITY OF CHILDREN UNDER 3 YEARS OF AGE.

17. In tape 17 the values from equations (1), (2), and (3) were set up. The deaths during the period 1909–1911 were added on the right of the adding machine and the corresponding number of children, or the equivalent generation, was obtained on the left. To obtain the values needed in equation (1) the deaths aged 0–1, D_0, for 1911, 1910, 1909, were added on the right side of the machine, and at the same time the number of births just above them in Table 12 were added on the left. To the left side was then added one-third of the first total in tape 16, 99997241, and a total taken.

To obtain the values needed in equation (2) the total just obtained on the left was added to the complement of the total on the right and to this was added the remainder (99994482) of the first total and one-half of the second total in tape 16. On the right side of the machine the deaths aged 1–2, D_1, in the calendar years 1911, 1910, and 1909 were set up and a total taken.

To obtain the values needed in equation (3) the total just obtained on the left was added to the complement of the total on the right, and to this one-half of the third total in tape 16 was added. On the right side of the machine the deaths 2–3, D_2, in the calendar years 1911, 1910, and 1909 were set up and a total taken. Then each total on the right was divided by the corresponding total on the left to obtain the rate of mortality at each age. The result to the nearest sixth decimal place was set up as a whole number under the heading $10^6 q_x$.

ORIGINAL STATISTICS FOR DETERMINING RATES OF MORTALITY AT AGES 7 YEARS AND OVER.

18. The original statistics, on which the life table for males in the state of New York, 1909–1911, was based, were obtained from the United States Life Tables, 1890, 1901, 1910, 1901–1910, page 450, Table 159. The populations in column 2 and the deaths in column 6 were summed in the quinquennial age groups 0–4, 5–9, 10–14, and so on through the group 95–99. The machine was split between banks 15–16 and 8–9, ages being entered in banks 16–17. Beginning with the age group 5–9, the populations were entered on the left side of the machine and the deaths on the right side, and a subtotal was taken after the group 95–99 was entered. To these subtotals the populations and deaths, respectively, 100 years of age and over, and the age groups 0–4, were added in order to check to the total populations and deaths as given in Table 159 mentioned above. The values in tape 18 are the $P^{x/x+4}_{1920}$ and the $P^{1919-20}_{x/x+4}$ required by equations (7) to (10), pages 33 and 34, to obtain the graduated values of L_x and $(3d)_x$ for $x=7$, 12, 17, and so on. These are the central ages of the quinquennial age groups 5–9, 10–14, 15–19, and so on.

APPLICATION OF EQUATIONS (7) TO (10) TO THE STATISTICS IN TAPE 18.

19. For convenience of reference equations (7) to (10) are given with subscripts for period 1909–1911.

$$10^3 L_7 = (-10+2)\ (P^{5/9}_{1910} - 2P^{10/14}_{1910} + P^{15/19}_{1910}) + 200 P^{5/9}_{1910} \qquad (9)$$

$$10^3 (3d)_7 = (-10+2)\ (D^{1909-11}_{5/9} - 2D^{1909-11}_{10/14} + D^{1909-11}_{15/19}) + 200 D^{1909-11}_{5/9} \qquad (10)$$

$$10^3 L_{x+2} = (-10+2)\ (P^{x-5/x-1}_{1910} - 2P^{x/x+4}_{1910} + P^{x+5/x+9}_{1910}) + 200 P^{x/x+4}_{1910} \qquad (7)$$

$$10^3 (3d)_{x+2} = (-10+2)\ (D^{1909-11}_{x-5/x-1} - 2D^{1909-11}_{x/x+4} + D^{1909-11}_{x+5/x+9}) + 200 D^{1909-11}_{x/x+4} \qquad (8)$$

It was found convenient to split the adding machine between banks 9 and 10 and to apply equations (9) and (7) to the numbers on the left of tape 18 in banks 10 to 17 of the adding machine while applying equations (10) and (8) to the numbers on the right of tape 18 in banks 1 to 9. Accordingly, the first numbers in tape 18 (405163 and 4710) were set up in corresponding places on the adding machine and beneath them the complements of the second set of numbers in tape 18 were repeated twice and then the third set added. The numbers now appearing at the base of the adding machine, 24737 and 3820, are the values of the quantities in the second parentheses of equations (9) and (10), and are really second differences but may be called the operands. Since these operands are to be operated on by $+2$ and -10, they were added in unit's place and their complements in ten's place. In accordance with the last expressions in equations (9) and (10), the first numbers in tape 18 were added twice in hundred's place and a total taken. The sum on the right, 80834704, is $1000 L_x$ and that on the left, 911440, is $1000(3d)_x$.

When 10 is substituted for x in equations (7) and (8), the left-hand members of the equations are $10^3 L_{12}$ and $10^3(3d)_{12}$, while the operands are the same as in equations (9) and (10). Accordingly, the values for these operands, 24737 and 3820, were repeated twice in unit's place and their complements added in ten's place, and the second set of numbers in tape 18, 396114 and 2855, are repeated twice in hundred's place and a total taken. When 15 is substituted for x in equations (7) and (8), the left-hand members of the equations are $10^3 L_{17}$ and $10^3(3d)_{17}$, while the first numbers in the operands are the 396114 and 2855 which appear in hundred's place just before the last total. To these are added the complements (repeated twice) of the numbers just below them in tape 18, 411802 and 4820, and then the fourth set of numbers in tape 18. The totals then appearing at the base of the adding machine, 36060 and 827, are set up in unit's place and their complements in ten's place in accordance with the operators $+2$ and -10, and to them are added the 411802 and 4820 in hundred's place (repeated twice), which are

RATES OF MORTALITY UNDER 3 YEARS OF AGE.

CALCULATION OF THE LIFE TABLE FOR MALES IN THE STATE OF NEW YORK: 1909-1911.

PHOTOGRAPHS OF ADDING MACHINE TAPES ON WHICH COMPUTATIONS WERE MADE.

16

x — δ_x and $k_x \delta_x$

	10535	101.09 #
	119050	1111.99
	129585	1213.08
	99870415	99870415
		3 | 99991723 #
0	$k_0\delta_0$ =	99997241 #
	$(1-k_0)\delta_0$ =	99994482 #
	1905	17.66
	4097	39.31
	11111	109.68
	115948	1064.65
	133061	1231.30
	99866939	99866939
		2 | 99990069 #
1	$\frac{1}{2}\delta_1$ =	99995035 #
	764	7.00
	1324	12.27
	2014	20.07
	4321	42.66
	10950	104.90
	111666	995.44
	131039	1182.34
	99868961	99868961
		99987195
		99990069
		2 | 199977264 #
2	$\frac{1}{2}[\delta_1+\delta_2]$ =	99988632

17

x — $E_x+k_x\delta_x$ — $10^6 q_x$ — D_x

x	$E_x+k_x\delta_x$	$10^6 q_x$	D_x
	119050		140.40 #
	115948		152.34
	111666		145.69
	99997241		
	343905		438.43 #
0		127486	
	343905		29.93
	99956157		34.01
	99994482		35.23
	99995035		
	289579		99.17 #
1		34246	
	289579		13.20
	99990083		15.45
	99988632		14.84
	268294		43.49 #
2		16210	

18

x	$P_{1910}^{x/x+4}$	$D_{x/x+4}^{1909-11}$
5	405163	47.10
10	396114	28.55
15	411802	48.20
20	463550	76.12
25	453622	89.29
30	399449	101.92
35	367773	123.58
40	312661	126.05
45	260805	131.89
50	216306	137.11
55	149105	131.17
60	115823	143.23
65	84802	142.53
70	56690	134.66
75	32248	113.03
80	15543	78.40
85	5680	41.86
90	1451	13.61
95	222	2.69
50	4148809	1710.99
	42	.57
	456206	623.42
50	4605057	2334.98 #

19

$10^3 L_x$	$10^3 (3d)_x$
405163	47.10
99603886	99971.45
99603886	99971.45
411802	48.20
24737	38.20
99752630	99618.00
40516300	4710.00
40516300	4710.00
80834704	3 | 9114.40 #
24737	38.20
24737	38.20
99752630	99618.00
39611400	2855.00
39611400	2855.00
79024904	1 | 5404.40 #
396114	28.55
99588198	99951.80
99588198	99951.80
463550	76.12
36060	8.27
99639400	99917.30
41180200	4820.00
41180200	4820.00
82071920	3 | 9573.84 #
411802	48.20
99536450	99923.88
99536450	99923.88
453622	89.29
99938324	99985.25
616760	147.50
46355000	7612.00
46355000	7612.00
93203408	3 | 15342.00 #

463550	76.12
99546378	99910.71
99546378	99910.71
399449	101.92
99955755	99999.46
442450	5.40
45362200	8929.00
45362200	8929.00
91078360	3 | 17862.32 #
453622	89.29
99600551	99898.08
99600551	99898.08
367773	123.58
22497	9.03
99775030	99909.70
39944900	10192.00
39944900	10192.00
79709824	3 | 20311.76 #
399449	101.92
99632227	99876.42
99632227	99876.42
312661	126.05
99976564	99980.81
234360	191.90
36777300	12358.00
36777300	12358.00
73742088	3 | 24869.52 #
367773	123.58
99687339	99873.95
99687339	99873.95
260805	131.89
3256	3.37
99967440	99966.30
31266100	12605.00
31266100	12605.00
62506152	3 | 25183.04 #
312661	126.05
99739195	99868.11
99739195	99868.11
216306	137.11
7357	99999.38
99926430	6.20
26080500	13189.00
26080500	13189.00
52102144	3 | 26382.96 #
260805	131.89
99783694	99862.89
99783694	99862.89
149105	131.17
99977298	99988.84
227020	111.60
21630600	13711.00
21630600	13711.00
43442816	3 | 27511.28 #
216306	137.11
99850895	99868.83
99850895	99868.83
115823	143.23
33919	18.00
99660810	99820.00
14910500	13117.00
14910500	13117.00
29549648	3 | 26090.00 #

the second numbers in the operands. After a total is taken, the 411802 and 4820 are entered in unit's place to begin the next computation. The operator soon learns this routine of repeating twice in hundred's place the second numbers in the operands, whose complements were repeated twice, and then so soon as a total is taken, starting the next set of computations with the same set of numbers, and the results can be obtained very rapidly by a careful machine operator without his understanding negative values, differencing, or decimals.

DETERMINATION OF NUMBER EXPOSED TO RISK OF DEATH TO OBTAIN RATES OF MORTALITY.

20. The rates of mortality were determined according to equations $q_x = d_x/(L_x + .5d_x)$. Since the deaths were for a 3-year period, as indicated by the symbols $(3l)_x$ and $(3d)_x$, and it was desired to obtain average annual rates, either the deaths had to be divided by three or the population multiplied by three. The latter method was found to be more convenient. Accordingly the above equation was written:

$$q_x = (3d)_x/[3L_x + \tfrac{1}{2}(3d)_x]. \qquad (23)$$

In tape 20 the values of the denominator, $3L_x + \frac{1}{2}(3d)_x$, were determined by adding to the totals on the left side of tape 19, repeated three times, one-half of the corresponding totals on the right side of tape 19.

21. In order to check the work from tapes 18 to 20, and for convenience in dividing, the totals in tape 20 were added in tape 21. These totals are the $10^3[3L_x + \frac{1}{2}(3d)_x]$ of equation (23).

22. Also the totals on the right side of tape 19 were added and fastened to the right side of the values in tape 21. They are the $(3d)_x$ of equation (23). Where the populations are small and the period is for two years instead of for three, so that only $2L_x + \frac{1}{2}(2d)_x$ is needed for the denominator in equation (23), it is often convenient to add these two sets of values on the same tape, the $10^3[2L_x + \frac{1}{2}(2d)_x]$ on the left side and the $(2d)_x$ on the right.

With the two tapes, 21 and 22, side by side, the operator performs the divisions indicated in equation (23), and enters the quotients to the nearest sixth decimal between them. Then they were cleared of fractions by entering them under the heading $10^6 q_x$.

23. Table 13 shows how the values in tape 18 enter into the totals in tape 19. In this table the values in tape 18 are represented by w_x at the top of the columns, and the totals in tape 19 by u_y in the left-hand margin. The coefficients of w_x in the equation for u_y are in the same line with u_y and each coefficient is in the same column with the w_x to which it belongs.

NUMBER EXPOSED TO RISK OF DEATH FOR ONE YEAR.

CALCULATION OF LIFE TABLE FOR MALES IN THE STATE OF NEW YORK: 1909-11.

PHOTOGRAPHS OF ADDING MACHINE TAPES UPON WHICH CALCULATIONS WERE MADE.

	*
149105	131.17
99884177	99856.77
99884177	99856.77
84802	142.53
2261	99987.24
99977390	127.60
11582300	14323.00
11582300	14323.00
23146512	$28748.08*
	*
	*
115823	143.23
99915198	99857.47
99915198	99857.47
56690	134.66
2909	99992.83
99970910	71.70
8480200	14253.00
8480200	14253.00
16937128	$28563.36*
	*
	*
84802	142.53
99943310	99865.34
99943310	99865.34
32248	113.03
3670	99986.24
99963300	137.60
5669000	13466.00
5669000	13466.00
11308640	$27042.08*
	*
	*
56690	134.66
99967752	99886.97
99967752	99886.97
15543	78.40
7737	99987.00
99922630	130.00
3224800	11303.00
3224800	11303.00
6387704	$22710.00*
	*
	*
32248	113.03
99984457	99921.60
99984457	99921.60
5680	41.86
6842	99998.09
99931580	19.10
1554300	7840.00
1554300	7840.00
3053864	$15695.28*
	*
	*
15543	78.40
99994320	99958.14
99994320	99958.14
1451	13.61
5634	8.29
99943660	99917.10
568000	4186.00
568000	4186.00
1090928	$ 8305.68*
	*
	*
5680	41.86
99998549	99986.39
99998549	99986.39
222	2.69
3000	17.33
99970000	99826.70
145100	1361.00
145100	1361.00
266200	$ 2583.36*

20 Additions to obtain

x	$10^9[3L_x+\frac{1}{2}(3d)_x]$
	*
	808347.04
	808347.04
	808347.04
	4557.20
7	2429598.32*
	*
	*
	790249.04
	790249.04
	790249.04
	2702.20
12	2373449.32*
	*
	*
	820719.20
	820719.20
	820719.20
	4786.92
17	2466944.52*
	*
	*
	932034.08
	932034.08
	932034.08
	7671.00
22	2803773.24*
	*
	*
	910783.60
	910783.60
	910783.60
	8931.16
27	2741281.96*
	*
	*
	797098.24
	797098.24
	797098.24
	10155.88
32	2401450.60*
	*
	*
	737420.88
	737420.88
	737420.88
	12434.76
37	2224697.40*
	*
	*
	625061.52
	625061.52
	625061.52
	12591.52
42	1887776.08*
	*
	*
	521021.44
	521021.44
	521021.44
	13191.48
47	1576255.80*
	*

	*
	434428.16
	434428.16
	434428.16
	13755.64
52	1317040.12*
	*
	*
	295496.48
	295496.48
	295496.48
	13045.00
57	899534.44*
	*
	*
	231465.12
	231465.12
	231465.12
	14374.04
62	708769.40*
	*
	*
	169371.28
	169371.28
	169371.28
	14281.68
67	522395.52*
	*
	*
	113086.40
	113086.40
	113086.40
	13521.04
72	352780.24*
	*
	*
	63877.04
	63877.04
	63877.04
	11355.00
77	202986.12*
	*
	*
	30538.64
	30538.64
	30538.64
	7847.64
82	99463.56*
	*
	*
	10909.28
	10909.28
	10909.28
	4152.84
87	36880.68*
	*
	*
	2662.00
	2662.00
	2662.00
	1291.68
92	9277.68*

TABLE 13.—DERIVATION OF FORMULA FOR CHECK ON WORK IN TAPES 18 TO 22.

COMPUTATION OF CHECK IS GIVEN IN TAPE 23.

This table shows the coefficients of the values in tape 18 in the equations for the totals in tape 19, derived according to equations (7) to (10), page 38. The values in tape 18 are represented by w_x at the head of the columns and the totals in tape 19 by u_y in the left-hand margin. Any number in the table is the coefficient of the w_x at the head of its column in the equation for the u_y in the left margin of its line.

	w_5	w_{10}	w_{15}	w_{20}	w_{25}	w_{30}	and so on to	w_{75}	w_{80}	w_{85}	w_{90}	w_{95}
u_7	200−8	+16	−8									
u_{12}	−8	200+16	−8									
u_{17}	....	−8	200+16	−8								
u_{22}	....	....	−8	200+16	−8							
u_{27}	....	....	....	−8	200+16	−8						
and so on to												
u_{82}	....	....	....	....	....	....	....	−8	200+16	−8		
u_{87}	....	....	....	....	....	....	....	....	−8	200+16	−8	
u_{92}	....	....	....	....	....	....	....	....	....	−8	200+16	−8
Total	200−16	200+24	200−8	200	200	200		200	200	200	200+8	−8

Thus 200 times either sum in tape 18, ages 5 to 95 (4148809 and 171099), lacks $-16w_5+24w_{10}-8w_{15}+8w_{90}-8w_{95}-200w_{95}$ of being equal to the sum of the corresponding totals in tape 19. This expression may be written as $(+2-10)\ (2w_5-3w_{10}+w_{15}-w_{90}+w_{95})-200w_{95}$. Then the sum of u_y for $y=7$ to $y=92$ is equal to the sum of 200 times the totals, ages 5 to 95, in tape 18 plus $(+2-10)\ (2w_5-3w_{10}+w_{15}-w_{90}+w_{95})-200w_{95}$. These additions are performed in tape 23, those for populations under tape 21, and those for deaths under tape 22. As in tape 19 the values of the operands were first obtained, and these were then added in unit's place and their complements in ten's place; then the complements of w_{95} were added once and the subtotals in tape 18 (4148809 and 171099) twice in hundred's place. A subtotal was then taken in the addition for populations, and this subtotal repeated twice and one-half the total of the deaths ($\frac{1}{2}\times 34129336$) added to it. As indicated by the symbols ① and ② to the right of the totals in tapes 21 and 22, respectively, and of those beneath in tape 23, the corresponding totals agree, indicating that the computations from tapes 18 to 23 are correct.

PROCESS OF OBTAINING THE log p_x NEEDED TO COMPUTE log ${}_5p_x$.

24. Formulas 13 to 16 for determining $\log {}_5p_x$ required log p_x. Accordingly the 10^6q_x in tape 24 were copied on the left of the machine and at the same time their complements, p_x, or in this case, $1{,}000{,}000-10^6q_x=10^6p_x$, were set up on the right. After each addition the totals should be found to be complementary as are the totals at the end of the tape. To indicate this agreement the operator adds the subtotal on the left of the machine to that on the right. The total should be 0 in the first six places and 21 in the next two places. The 21 shows the operator how many terms he has set down.

25. Bauschinger and Peters eight-place logarithmic tables were used to obtain log p_x. The mantissa of the logarithm of the first five digits of the p_x could be read directly from the book, and this was set up on the adding machine. Then the operator looked up the P. P. (proportional part) which corresponded to the sixth figure in p_x and added it to the mantissa of the first five digits, and took a total. Since the characteristics of all these log p_x's were -1, the characteristics are omitted here and in the tapes that follow until tape 37, the additions for log l_x. Also the decimal point is omitted. Accordingly 10^8 (log p_x+1), is put in the headings of tapes 25 to 36, but in the discussion of the tapes simply log p_x is used.

To condense the work, the machine was split between banks 9–10, and the mantissas for two consecutive logarithms were set up side by side. That is, after the two parts of the mantissa of the first logarithm had been entered on the left of the adding machine, the platen was rolled back two places and the two parts of the mantissa of the second log p_x were added before a total was taken. Putting the logarithms on a tape in this form is of great convenience to the comparer and also tends to increase the accuracy of the computer.

EXTENSION OF THE SERIES OF log p_x TO A VERY OLD AGE.

26–30. As explained in section 11, the mantissas of the last seven values of log p_x in tape 25 were copied on a separate tape and differenced four times in tapes 27 to 30. This includes the logarithms of p_x from $x=62$ to $x=92$. The method of making these tapes was as follows: The first value in the tape for $\Delta^n(\log p_x+1)$ was set up at the beginning of the tape for $\Delta^{n+1}(\log p_x+1)$, and then the operator mentally subtracted the first value in the $\Delta^n(\log p_x+1)$ tape from the one next below it and added the remainder under the first value which was set up at the beginning

RATES OF MORTALITY AND LOGARITHMS OF THE PROBABILITY OF LIVING ONE YEAR.

CALCULATION OF LIFE TABLE FOR MALES IN THE STATE OF NEW YORK: 1909–1911.

PHOTOGRAPHS OF ADDING MACHINE TAPES UPON WHICH CALCULATIONS WERE MADE.

21

x	$10^3[3L_x+\frac{1}{2}(3d)_x]$
7	242959832
12	237344932
17	246694452
22	280377324
27	274128196
32	240145060
37	222469740
42	188777608
47	157625580
52	131704012
57	89953444
62	70876940
67	52239552
72	35278024
77	20298612
82	9946356
87	3688068
92	927768
	2505435500* (1)

22

$10^6 q_x$	$(3d)_x$
3751	911440
2277	540440
3881	957384
5472	1534200
6516	1786232
8458	2031176
11179	2486952
13340	2518304
16738	2638296
20889	2751128
29004	2609000
40561	2874808
54678	2856336
76654	2704208
111880	2271000
157799	1569528
225204	830568
278449	258336
	34129336* (2)

23

Checks on Computations Tapes 18 to 22

405163	4710
405163	4710
9999960388 6	9999999971 45
9999960388 6	9999999971 45
9999960388 6	9999999971 45
411802	4820
9999999985 49	9999999986 39
222	269
32557	4583
9999996744 30	9999999541 70
9999999778 00	9999999731 00
9999999778 00	9999999731 00
4148809 00	1710 99 00
4148809 00	1710 99 00
829456944 8	
829456944	
829456944	
17064668	
2505435500* (1)	34129336* (2)

Operands

24

x	$10^6 q_x$	$10^6 p_x$
0	127486	872514
1	34246	965754
2	16210	983790
7	3751	996249
12	2277	997723
17	3881	996119
22	5472	994528
27	6516	993484
32	8458	991542
37	11179	988821
42	13340	986660
47	16738	983262
52	20889	979111
57	29004	970996
62	40561	959439
67	54678	945322
72	76654	923346
77	111880	888120
82	157799	842201
87	225204	774796
92	278449	721551
	1244672	19755328 8
		1244672
	1244672	21000000*

25

$10^8(\log p_x+1)$

94077041	98486472
199	180
94077240	98486652*
99290240	99836398
	392
99290840	99836790*
99900868	99830730
131	392
99900999	99831122*
99761352	99715913
350	175
99761702	99716088*
99631024	99511724
88	44
99631112	99511768*
99416752	99266837
	88
99416752	99266925*
99083149	98721476
44	268
99083193	98721744*
98201329	97557885
408	92
98201737	97557977*
96536165	94847165
282	
96536447	94847165*
92541524	88918401
52	336
92541576	88918737*
85826643	
60	
85826703*	

26

x	$10^8(\log p_x+1)$
62	98201737
67	97557977
72	96536447
77	94847165
82	92541576
87	88918737
92	~~85826703~~ (3)
	654430342*

27

98201737

x	$10^8\Delta[\log p_x+1]$
62	9999993562 40
67	9999989784 70
72	9999983107 18
77	9999976944 11
82	9999963771 61
87	~~9999969079 66~~ (4)
	85826703* (3)

28

9999993562 40

x	$10^8\Delta^2[\log p_x+1]$
62	9999996222 30
67	9999993322 48
72	9999993836 93
77	9999986827 50
82	~~5308 05~~ (5)
	9999969079 66* (4)

of the $\Delta^{n+1}(\log\ p_x+1)$ tape. If the subtraction is correct, the second value appears through the glass at the base of the adding machine. This is in accordance with the equation $\Delta^n u_x+\Delta^{n+1}u_x=\Delta^n u_{x+5}$. If the first value in the $\Delta^n(\log\ p_x+1)$ is larger than the second, the subtraction is made as though the second value had been increased by 10^{12}, or whatever multiple of 10 is necessary to carry it beyond the split. This process of differencing is described fully in the United States Life Tables: 1890, 1901, 1910, 1901–1910, page 374, section 149.

31. An examination of tape 30 shows that these fourth differences are very rough. Either $\Delta^4\log\ p_{62}$ or $\Delta^4\log\ p_{72}$, if used as a constant $\Delta^4\log\ p_x$ for all older ages, would give the *greatest* probability of living at the *oldest* age. Only $\Delta^4\log\ p_{67}$ would produce reasonable results, if it were used as a constant $\Delta^4\log\ p_x$ for ages older than 67. Accordingly this assumption was made, and $\Delta^4\log\ p_{67}$ was added to the $\Delta^3\log\ p_{72}$ six times, a subtotal being taken after the first five additions and a total at the end. Tape 31 shows this work. The first subtotal is used as $\Delta^3\log\ p_{77}$ in place of 1,848,055 which produced such an irregular $\Delta^4\log\ p_{72}$. The other five subtotals are used as $\Delta^3\log\ p_{82}$ to $\Delta^3\log\ p_{102}$.

32. In tape 32 the five subtotals and the total in tape 31 were added to $\Delta^2\log\ p_{77}$, a subtotal being taken after each addition until the last when a total was taken. These subtotals and total serve as $\Delta^2\log\ p_x$ from $x=82$ to $x=107$.

33–34. In the same way the subtotals and the total in tape 32 were added to $\Delta\log\ p_{82}$ to obtain $\Delta\log\ p_x$ for $x=87$ to $x=112$ in tape 33, and in tape 34 these new values of $\Delta\log\ p_x$ were added to $\log\ p_{87}$ to obtain $\log\ p_{92}$ to $\log\ p_{117}$. As stated in section 11, these values of $\log\ p_x$ to a very old age were used to determine $\log\ {}_5p_x$ to ages old enough to reduce the radix of 100,000 to less than 0.5 or practically 0.

PROCESS OF OBTAINING $\log\ {}_5p_x$ NEEDED TO COMPUTE l_x AT FIVE YEAR INTERVALS.

35. The $10^8[\log\ p_x+1]$ obtained in tape 25 were copied in tape 35, except the last, for age 92, which was replaced by its estimated value in tape 34 The values in tape 25 were then followed by the other estimated $10^8[\log\ p_x+1]$ in tape 34. Since equations (13) to (16) and so on, page 35, do not require the logarithms of p_0 and p_1, they were added separately at the beginning of tape 35 and a total taken. The addition was begun with $\log\ p_2$ and continued through $\log\ p_{117}$.

36. In obtaining the value of $10^9\log\ {}_5p_2$ according to equation (13), ten times the value of the operand was obtained first. $10^8(\log\ p_2+1)$ in tape 35 was set up in ten's place, $10^8(\log\ p_7+1)$ repeated four times in ten's place, and the complement of $10^8(\log\ p_{12}+1)$ repeated twice in ten's place. This gave ten times the operand, which was read through the glass of the machine and set up again, and then one-tenth of its complement added three times. Then, in accordance with the other terms in equation (13), the complement of $10^8(\log\ p_2+1)$ was added in unit's place, that of $10^8(\log\ p_7+1)$ in ten's place, and $10^8(\log\ p_{17}+1)$ was added in ten's place, and a total taken. This total is $10^9(\log_5\ p_2+5)$.

To obtain the value for $10^9\log\ {}_5p_7$ according to equation (14) $10^8[\log\ p_7+1]$ in tape 35 was added to $10^8[\log\ p_{12}+1]$ and a subtotal taken. Then the subtotal was set up and repeated three times in unit's place and set up again and repeated twice in ten's place, so that the total on the machine at the end of this step in the work may be represented by the expression $24[10^8(\log\ p_7+1+\log\ p_{12}+1)]$. This is in accordance with the first term on the right of equation (14). Then in accordance with the next three terms, $10^8(\log\ p_{12}+1)$ was set up in ten's place, the complement of $10^8(\log\ p_{17}+1)$ was set up in ten's place and $10^8[\log\ p_{22}+1]$ is repeated twice in unit's place, giving as a total,

$$10^8[24(\log\ p_7+\log\ p_{12})+10\log\ p_{12}-10\log\ p_{17}+2\log\ p_{22}\\ +(48+10-10+2)]=10^8\,[24(\log\ p_7+\log\ p_{12})\\ +10\log\ p_{12}-10\log\ p_{17}+2\log\ p_{22}]+5(10^9).$$

In other words the result obtained is $10^9\log\ {}_5p_7+5(10^9)$.

In the formulas (13) to (16) it will be noted that only four consecutive values of $\log\ p_x$ are used in each period. In this connection it was found convenient to use as a marker a cardboard with a rectangular opening cut in it just wide enough to allow four of the values on tape 35 to be seen.

Since the same ages appear in equation (15) as in equation (14), the cardboard was not moved, but the four values were added again in a different way. It will be noted that the values for ages 7 and 22, the first and last of the four values appearing in the opening of the cardboard, are in the first expression on the right of equation (15) with the coefficient -1, while the two middle values have the coefficient $+11$ in this expression. Accordingly the first and last values in the opening of the cardboard were added first and a total taken. Then the two middle values were added and their sum, appearing at the base of the adding machine, was set up in ten's place. This gave $10^8\,[11\,(\log p_{12}+1+\log p_{17}+1)]$. To this the complement of the first two values were added, giving 10^8 times the value of the expression—

$$-\log p_7-1+11(\log p_{12}+1+\log p_{17}+1)-\log p_{22}-1\\ =20+[-\log p_7+11(\log p_{12}+\log\ p_{17})-\log p_{22}]$$

The expression in brackets is the same as that in equation (15). Since twice this expression is required, the sum appearing at the base of the adding machine was

PROBABILITIES OF LIVING ONE YEAR AT VERY OLD AGES.

CALCULATION OF LIFE TABLE FOR MALES IN THE STATE OF NEW YORK: 1909-1911.

PHOTOGRAPHS OF ADDING MACHINE TAPES UPON WHICH CALCULATIONS WERE MADE.

29

x		$10^8\Delta^3[\log p_x+1]$
		9999996222.30 #
62		9999997100.18
67		514.45
72		9999992990.57
77		~~18480.55~~ (6)
	3	5308.05 # (5)

30

x		$10^8\Delta^4[\log p_x+1]$
		9999997100.18
62		3414.27
67		9999992476.12
72		~~25489.98~~
	2	18480.55 # (6)

31

x		$10^8\Delta^3[\log p_x+1]$
72		9999992476.12
		9999992990.57
77	1	9999985466.69
		9999992476.12
82	2	9999977942.81
		9999992476.12
87	3	9999970418.93
		9999992476.12
92	4	9999962895.05
		9999992476.12
97	5	9999955371.17
		9999992476.12
102	6	9999947847.29 #

32

x		$10^8\Delta^2[\log p_x+1]$
77		9999986827.50
		9999985466.69
82	1	9999972294.19
		9999977942.81
87	2	9999950237.00
		9999970418.93
92	3	9999920655.93
		9999962895.05
97	4	9999883550.98
		9999955371.17
102	5	9999838922.15
		9999947847.29
107	6	9999786769.44 #

33

x		$10^8\Delta[\log p_x+1]$
82		9999963771.61
		9999972294.19
87	1	9999936065.80
		9999950237.00
92	2	9999886302.80
		9999920655.93
97	3	9999806958.73
		9999883550.98
102	4	9999690509.71
		9999838922.15
107	5	9999529431.86
		9999786769.44
112	6	9999316201.30 #

34

x		$10^8[\log p_x+1]$
87		889187.37
		9999936065.80
92	1	825253.17
		9999886302.80
97	2	711555.97
		9999806958.73
102	3	518514.70
		9999690509.71
107	4	209024.41
		9999529431.86
112	4	9999738456.27
		9999316201.30
117	5	9999054657.57 #

35

x		$10^8[\log p_x+1]$	
0		940772.40	a
1		984866.52	b
		1925638.92 #	
2		992902.40	c
7		998367.90	d
12		999009.99	e
17		998311.22	f
22		997617.02	g
27		997160.88	
32		996311.12	
37		995117.68	
42		994167.52	
47		992669.25	
52		990831.93	
57		987217.44	
62		982017.37	
67		975579.77	
72		965364.47	
77		948471.65	
82		925415.76	
87		889187.37	
92		825253.17	
97		711555.97	
102		518514.70	
107		209024.41	x
112		9999738456.27	y
117		9999054657.57	z
	2	18683182.83 #	

36

Computations of $10^9[\log {}_5p_x+5]$

x		
		9929024.00
		9983679.00
		9983679.00
		9983679.00
		9983679.00
		9990009900.10
		9990009900.10
		29883540.20
		9997011645.98
		9997011645.98
		9997011645.98
		9999007097.60
		9990016321.00
		9983112.20
2	1	49808549.14 #
		998367.90
		999009.99
		1997377.89
		1997377.89
		1997377.89
		1997377.89
		19973778.90
		19973778.90
		999009.90
		9990016887.80
		997617.02
		997617.02
7	1	49939291.10 #
		998367.90
		997617.02
		1995984.92 #
		999009.99
		998311.22
		19973212.10
		9998004015.08
		19974548.39
		999009.90
12	1	49939196.68 #
		999009.99
		997160.88
		1996170.87 #
		998311.22
		997617.02
		19959282.40
		9998003829.13
		19959039.77
		9983112.20
17	1	49901191.74 #
		998311.22
		996311.12
		1994622.34 #
		997617.02
		997160.88
		19947779.00
		9998005377.66
		19947934.56
		9976170.20
22	1	49872039.32 #

set up again. In accordance with the last term in equation (15) the second value in the opening of the cardboard was added in ten's place and a total taken. The result is:

$$10^8\{2[20-\log p_7+11(\log p_{12}+\log p_{17})-\log p_{22}] +10(\log p_{12}+1)\}$$
$$=10^8\{2[-\log p_7+11(\log p_{12}+\log p_{17})-\log p_{22}] +10\log p_{12}\}+5(10^9)$$
$$=10^9\log {}_5p_{12}+5(10^9).$$

Then the cardboard was moved down one space and equation (16) applied to the next four consecutive values. It will be noted that equation (16) is the same general equation as (15). Hence the first and last values appearing in the cardboard were added and a total taken. Next the second and third values were added, their sum, appearing at the base of the adding machine, was set up in ten's place; the complement of the sum of the first and fourth just above was added; the sum appearing at the base of the adding machine was set up, and finally the second value in the opening was added in ten's place and a total taken. For reasons similar to the above, it will be found that this total is $10^9\log {}_5p_{17}+5(10^9)$. This same process was repeated on each four consecutive values in tape 35. These totals in tape 36 furnish the $\log {}_5p_x$ needed to obtain $\log l_x$ at every fifth year of age. The $\log {}_5p_x$ are in the following form: $10^9\log {}_5p_x+5(10^9)$.

$\log l_x$ AT EVERY FIFTH YEAR OF AGE.

37. Logarithms of l_x at every fifth year of age were obtained in tape 39 by adding to the logarithm of the radix $\log p_0$ and $\log p_1$ and then of each consecutive $\log {}_5p_x$, and taking a subtotal after each. Since the totals obtained in tape 36 are multiples of 10^9, the decimal point comes between banks 9 and 10 of the machine and may be indicated by a vertical line drawn between these banks. The radix is taken as 100,000, and since its logarithm is 5, this figure was added in the tenth bank of the adding machine.

Log p_0 is given in tape 35 as $10^8(\log p_0+1)$, and multiplying this expression by 10 changes it to $10^9\log p_0+10^9$. Accordingly 94,077,240, the first number in tape 35, was entered in ten's place. To remove the 10^9, 9's were set up from bank 10 to the split in the machine between banks 12 and 13, and a subtotal taken. This subtotal is $\log l_1$. In the same way $10^8(\log p_1+1)$, the second number in tape 35, was set up in ten's place, with 9's from bank 10 to bank 12. The subtotal taken here is $\log l_2$.

From age 2 the l_x are required at five-year intervals. Accordingly, $10^9\log {}_5p_2+5(10)^9$, the first total in tape 36, or 4,980,854,914, was added. To remove the $5(10)^9$, 5 was subtracted from the tenth bank of this first total, leaving 999 from banks 10 to 12 instead of 4 in bank 10. The subtotal taken here is $\log l_7$.

In this way each of the totals in tape 36 was added after 5 had been subtracted from the number in the tenth bank of the total, and a subtotal was taken after each addition. Since 4 is the number in the tenth bank of the totals in tape 36 from age 2 to 87, 999 is added in banks 10 to 12 for all these totals. The totals for age 92 and 97 contain 3 in the tenth bank, while those for ages 102 and 107 contain 2 and 0, respectively. Accordingly, for these four ages the numbers added in banks 10 to 12 in tape 37, were 998, 998, 997, 995, respectively.

Thus the series of subtotals in tape 37 are the logarithms of l_x. Whenever these subtotals became less than 999|698,000,000, which is log 0.5 on the adding machine tape, the remaining totals in tape 36 were added in without taking a subtotal, since all values in for l_x less than 0.5 were called 0.

Since 10^9 was subtracted from ten times each of the first two values in tape 35 before adding them in tape 37, and $5(10^9)$ was subtracted from each of the totals in tape 36 before they were added in tape 37, the total thus far obtained in tape 37 does not equal ten times the first two terms in tape 35 plus the totals in tape 36. Since there are always 22 totals in tape 36 and $5(10^9)$ was added at beginning of tape 37, this difference is $10^9(-5+2+5\times 22)=107(10^9)$. Therefore, for checking purposes 107 was added in banks 10 and 12 of tape 37 before the final total was taken. This final total is then ten times the first two values in tape 35 plus the totals in tape 36.

After this total had been checked, the antilogarithms of the subtotals in tape 37 were looked up in Bauschinger and Peters' logarithm tables and entered to the nearest integer to the left of the subtotal.

38. A check on the work in tapes 35 to 37 is derived in Table 14. In this table letters represent the $\log p_x$ for the values of x given just above them, and any number in the table is the coefficient of the $\log p_x$ at the top of its column in the equation for the $\log {}_5p_x$ on the left margin of the table in the same line with this number. These coefficients are taken from equations (13) to (16), and so on, page 35.

PROBABILITY OF LIVING FIVE YEARS.

CALCULATION OF LIFE TABLE FOR MALES IN THE STATE OF NEW YORK: 1909-1911.

PHOTOGRAPHS OF ADDING MACHINE TAPES UPON WHICH CALCULATIONS WERE MADE.

36 *Continued*

Computations of $10^9[\log {}_5p_x+5]$

```
                    #
      997617.02
      995117.68

     1992734.70 #
      997160.88
      996311.12
     1993472000
   99980072653.0
     1993545730
      997160880

27  1   4984252340 #
                    #
                    #
                    #
      997160.88
      994167.52

     1991328.40 #
      996311.12
      995117.68
     1991428800
   99980086716.0
     1991438840
      996311120

32  1   4979188800 #
                    #
                    #
                    #
      996311.12
      992669.25

     1988980.37 #
      995117.68
      994167.52
     1989285200
   99980110196.3
     1989315683
      995117680

37  1   4973749046 #
                    #
                    #
                    #
      995117.68
      990831.93

     1985949.61 #
      994167.52
      992669.25
     1986836770
   99980140503.9
     1986925486
      994167520

42  1   4968018492 #
                    #
                    #
                    #
      994167.52
      987217.44

     1981384.96 #
      992669.25
      990831.93
     1983501180
   99980186150.4
     1983712802
      992669250

47  1   4960094854 #
```

```
                    #
      992669.25
      982017.37

     1974686.62 #
      990831.93
      987217.44
     1978049370
   99980253133.8
     1978385645
      990831930

52  1   4947603220 #
                    #
                    #
                    #
      990831.93
      975579.77

     1966411.70 #
      987217.44
      982017.37
     1969234810
   99980335883.0
     1969517121
      987217440

57  1   4926251682 #
                    #
                    #
                    #
      987217.44
      965364.47

     1952581.91 #
      982017.37
      975579.77
     1957597140
   99980474180.9
     1958098663
      982017370

62  1   4898214696 #
                    #
                    #
                    #
      982017.37
      948471.65

     1930489.02 #
      975579.77
      965364.47
     1940944240
   99980695109.8
     1941989762
      975579770

67  1   4859559294 #
                    #
                    #
                    #
      975579.77
      925415.76

     1900995.53 #
      965364.47
      948471.65
     1913836120
   99980990044.7
     1915120179
      965364470

72  1   4795604828 #
```

```
                    #
      965364.47
      889187.37

     1854551.84 #
      948471.65
      925415.76
     1873887410
   99981454481.6
     1875820967
      948471650

77  1   4700113584 #
                    #
                    #
                    #
      948471.65
      825253.17

     1773724.82 #
      925415.76
      889187.37
     1814603130
   99982262751.8
     1818690961
      925415760

82  1   4562797682 #
                    #
                    #
                    #
      925415.76
      711555.97

     1636971.73 #
      889187.37
      825253.17
     1714440540
   99983630282.7
     1722187421
      889187370

87  1   4333562212 #
                    #
                    #
                    #
      889187.37
      518514.70

     1407702.07 #
      825253.17
      711555.97
     1536809140
   99985922979.3
     1549719847
      825253170

92  1   3924692864 #
                    #
                    #
                    #
      825253.17
      209024.41

     1034277.58 #
      711555.97
      518514.70
     1230070670
   99989657224.2
     1249649979
      711555970

97  1   3210855928 #
```

TABLE 14.—DERIVATION OF FORMULA FOR CHECK ON WORK IN TAPES 35 TO 37.

COMPUTATION OF CHECK IS GIVEN IN TAPE 38.

In this table letters represent the log p_x for values of x given just above the letters, and any number in the table is the coefficient of the log p_x at the top of its column in the equation for the log ${}_5p_x$ on the left margin of the table in the same line with this number. These coefficients are taken from equations (13) to (16), and so on, page 35.

10 log ${}_5p_x$ for x equals—	log p_x for x equals—																
	0	1	2	7	12	17	22	27	32	37	and so on to	92	97	102	107	112	117
	a	b	c	d	e	f	g	h	i	j		u	v	w	x	y	z
2	.....	.....	+16	+58	−34	+10											
7	.....	.....	.....	+24	+34	−10	+ 2										
12	.....	.....	.....	− 2	+32	+22	− 2										
17	.....	.....	.....	.....	− 2	+32	+22	− 2									
22	.....	.....	.....	.....	.....	− 2	+32	+22	− 2								
27	.....	.....	.....	.....	.....	.....	− 2	+32	+22	− 2							
32	.....	.....	.....	.....	.....	.....	.....	− 2	+32	+22							
37	.....	.....	.....	.....	.....	.....	.....	.....	− 2	+32							
42	.....	.....	.....	.....	.....	.....	.....	.....	.....	− 2							
and so on to																	
82	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	− 2					
87	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	+22	− 2				
92	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	+32	+22	− 2			
97	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	− 2	+32	+22	− 2		
102	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	− 2	+32	+22	− 2	.....
107	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	− 2	+32	+22	− 2
Total, $10\sum_{x=2}^{x=107} \log {}_5p_x$	.....	.....	+16	+80	+30	+52	+52	+50	+50	+50	and so on to	+50	+50	+50	+52	+20	− 2
$50\sum_{x=2}^{x=117} \log p_x$	.....	.....	+50	+50	+50	+50	+50	+50	+50	+50		+50	+50	+50	+50	+50	+50
$50\sum_{x=2}^{x=117} \log p_x - 10\sum_{x=2}^{x=107} \log {}_5p_x$			+34 =30+4	−30	+20	− 2	−2	0	0	0	.....	0	0	0	− 2	+30	+52= 2(30−4)

Therefore to reduce the sum of the totals in tape 36 to 50 times the second total in tape 35, ten times this sum must be increased by:

$(30+4)c-30d+20e-2f-2g \quad . \quad . \quad . \quad . \quad -2x+30y+2(30-4)z,$

or: $30(c-d+y+2z)+2(2c+10e-f-g-x-4z).$

This formula was reduced to the following form, convenient for computation upon the adding machine:

$2[2(c-2z)+10e-f-g-x]+3[10(c-d+y+2z)].$

Accordingly, the complete expression for the check on tapes 36–37 is:

I. Add c to complement of z repeated twice, and repeat total seen at the base of the machine. To this add e in ten's place and the complements of f, g, and x in unit's place and again repeat the total now seen at the base of the adding machine. Then clear machine.

II. Add $c-d+y+2z$ in ten's place and then set up the total seen in the glass at the base of the adding machine, repeating this total twice. To this add the total obtained in I just above, indicated by symbol ★, and then the total in tape 37.

III. Set up the second total in tape 35 and repeat 5 times; add to it, in ten's place, the first total in that tape, and take a total. The totals of II and III should agree. They are designated by the mark ⑦ to the right of each total.

Before starting his check the computer puts the letters a, b, c, d, e, f, g, and x, y, and z in the right margin of tape 35 to aid him in following the rule for the check. To preserve the first part of II, should his totals in II and III not agree, the operator takes a subtotal before adding the total in I.

39. No check was provided for the l_x determined by finding the antilogarithms of the subtotals in tape 37 except to compare them with the duplicate work. When this was done these l_x in pencil on tape 37 were added in tape 39. This put the l_x column in a more convenient form for deriving from it the $N'_{x:\overline{5}|}$ according to equations (17) to (20), page 35.

DETERMINATION OF $N'_{x:\overline{5}|}$ FROM l_x AND OF N'_x FROM $N'_{x:\overline{5}|}$.

40. Equations (17) to (20) are the same general equations as (13) to (16) and accordingly the same general method was used in computing the $N'_{x:\overline{5}|}$ from the l_x as was used in computing the log ${}_5p_x$ from the log p_x. The same cardboard was used to mark off the l_x to which the equation was being applied, and the addition was begun with the third number on the tape; the first two may be separated by a horizontal line. The method of computing $N'_{2:\overline{5}|}$ is identical with that of computing log ${}_5p_2$, but this is the only value of $N'_{x:\overline{5}|}$ obtained by the irregular formula.

However, since the l_x are much smaller numbers than the log p_x, it was found more convenient to add the first and last values appearing in the opening of the cardboard on the right of the adding machine and

NUMBER OF SURVIVORS.

CALCULATION OF LIFE TABLE FOR MALES IN THE STATE OF NEW YORK: 1909–1911.

PHOTOGRAPHS OF ADDING MACHINE TAPES UPON WHICH CALCULATIONS WERE MADE.

```
                          #
             71155597
      9999738456 27

  1          45001224 #
             518514.70
             209024.41
           7275391 10
      999954998 7.76
           755291797
           5185147 00

102 1      20290982 94 #
                          #
                          #
                          #
             518514.70
      9999054657 57

      9999573172 27 #
             209024 41
      9999738456 27
      9999474806 80
             426827.73
      9999849115 21
           2090244 10

107 3      178847 452 #
```

37

$10^9 \log l_x$

```
                                    #
100,000        5
            999940772400
87,251    1     494077240 0 S
            999984866520
84,263    2     492563892 0 S
            99998085491 4
80,629    3     490649383 4 S
            999993929110
79,510    4     490042294 4 S
            99999391966 8
78,405    5     489434261 2 S
            99999011917 4·
76,641    6     488446178 6 S
            999987203932
74,416    7     487166571 8 S
            999984252340
71,766    8     485591805 8 S
            999979188800
68,408    9     483510685 8 S
            999973749046
64,396   10     480885590 4 S
            999968018492
59,824   11     477687439 6 S
            99996009485 4
54,572   12     473696925 0 S
            999947603220
48,370   13     468457247 0 S
            999926251682
         14     ~~461082415 2 S~~
```

```
40,815   14     461082415 2 S
            99989821469 6
32,288   15     450903884 8 S
            99985955929 4
23,367   16     436859814 2 S
            999795604828
14,595   17     416420297 0 S
            99970011358 4
7,317    18     386431655 4 S
            999562797682
2,674    19     342711423 6 S
            999333562212
576      20     276067644 8 S
            99892469286 4
48       21     168536931 2 S
            998210855928
1        21 999896225240 S
            997  2909829 4
0        22 996925323534 S
            995178847452
            107
         24     99104170986 #
```

38

Check on tapes 35 to 37

```
                                      #
               99290240  c
               94534243 -z
               94534243 -z
              288358726 c-2z
              999009990 10e
           999900168878 -f
           999900238298 -g
           999979097559 -x
             1355232177
    3        2710464354 ★    #
                              #
                              #
             992902400 10c
          999001632100 -10d
          999738456270 10y
          999054657570 10z
          999054657570 10z
          997842305910
          997842305910
            2710464354 ★
Total-Tape 37 → 99104170986
    6      95341553070 (7)  #
                             #
                             #
                             #
                             #
Second Total Tape 35 → 186831828 30
                       186831828 30
                       186831828 30
                       186831828 30
                       186831828 30

                       934159141 50 S
First Total Tape 35 →  192563892 0
                                   (7)
                       953415530 70 #
```

39

x	l_x
0	1000.00
1	872.51
2	842.63
7	806.29
12	795.10
17	784.05
22	766.41
27	744.16
32	717.66
37	684.08
42	643.96
47	598.24
52	545.72
57	483.70
62	408.15
67	322.88
72	233.67
77	145.95
82	73.17
87	26.74
92	5.76
97	.48
102	1
	1150132 #

40

Computations of $N'_{x:\overline{5}|}$

```
                                 #
                       842.63
                       806.29

                      1648.92 S
                      1648.92
                      1648.92
                      1648.92
                     16489.20
                     16489.20
                       806290
                   9992049.00
                       784.05
                       784.05

           2         41254.08 #
                                 #
                                 #
      80629            842.63
      79510            784.05
    1601390
   99837332
    1598861
     806290
                  7
    4004012           1626.68 #
                                 #
                                 #
      79510            806.29
      78405            766.41
    1579150
   99842730
    1579795
     795100
                 12
    3954690           1572.70 #
                                 #
                                 #
      78405            795.10
      76641            744.16
    1550460
   99846074
    1551580
     784050
                 17
    3887210           1539.26 #
```

the second and third on the left, the machine being split between banks 9–10. Accordingly the first value in the opening was set up on the right and the second on the left and the adding machine lever struck. Then the third was set up on the left and the fourth on the right and the lever again touched. The total seen at the base of the machine on the left was set up in ten's place, the complement of the total seen through the glass on the right was added to this, and the total then seen through the glass on the left was added. Then ten times the second number appearing in the opening of the cardboard was added on the left and a total taken. This process was continued until a total had been obtained from each group of four consecutive l_x. A total was then obtained from the last three l_x according to equation (21), page 35, which gives $10N'_{(w-5):\overline{5}|}$. Since $l_w=1$, $N'_{w:\overline{5}|}$ was taken simply as 1. That is, $\sum_{x=w+1}^{x=w+4} l_x$ is zero in this table.

According to equations (17) to (21) the totals in tape 40 are $10N'_{x:\overline{5}|}$.

41. Then to obtain N'_x these $10N'_{x:\overline{5}|}$ were added, beginning with $10N'_{w:\overline{5}|}=10$ and taking a subtotal after each addition. After $10N'_{2:\overline{5}|}$ had been added and a subtotal taken, the $10N'_{1:\overline{1}|}$ and then the $10N'_{0:\overline{1}|}$ ($=1{,}000{,}000$) were added, a subtotal being taken after addition of the first and a total after addition of the last.

For the benefit of the reader the l_x were copied on tape 41 to the left of the N'_x for the same age. Thus, the dividends of equation (22), page 35, are given on the right side of tape 41 and the divisors in the center and the ages on the left margin of the tape. To aid the computer a vertical line between the first and second banks marks the decimal point in these N'_x.

42. The check on the work in tapes 39 to 41 is derived in Table 15. As in Table 14 any number in the table is the coefficient of the l_x at the top of its column in the equation for the $N'_{x:\overline{5}|}$ on the left margin of the page in the same line with the number. These equations for $N'_{x:\overline{5}|}$ are (17) to (21), page 35.

TABLE 15.—DERIVATION OF CHECK ON WORK IN TAPES 39 TO 41.

COMPUTATION OF CHECK IS GIVEN IN TAPE 42.

This table shows equations for $10\,N'_{x:\overline{5}|}$ in terms of l_x to l_{x+4} according to equations (17) to (21), page 35; $N'_{0:\overline{1}|}=l_0$ and $N'_{1:\overline{1}|}=l_1$.

Any number in the table from columns l_2 to $l_{w+5}=0$ is the coefficient of the l_x at the top of its column in the equation for the $10\,N'_{x:\overline{5}|}$ in first column in same line with the number.

$10N'_{x:\overline{y}|}$	l_0	l_1	l_2	l_7	l_{12}	l_{17}	l_{22}	l_{27}	and so on to	l_{w-20}	l_{w-15}	l_{w-10}	l_{w-5}	l_w	$l_{w+5}=0$	
$10N'_{0:\overline{1}|}$	+10															
$10N'_{1:\overline{1}|}$		+10														
$10N'_{x:\overline{5}|}$ for x equals—																
2			+24	+34	−10	+ 2										
7			− 2	+32	+22	− 2										
12				− 2	+32	+22	− 2									
17					− 2	+32	+22	− 2								
22						− 2	+32	+22								
27							− 2	+32								
32								− 2								
37																
And so on to																
w–30										− 2						
w–25										+22	− 2					
w–20										+32	+22	− 2				
w–15										− 2	+32	+22	− 2			
w–10											− 2	+32	+22	− 2		
w–5												− 2	+32	+22	− 2	
w														+10		$+10\sum_{x=w+1}^{x=w+4} l_x$
Total $10N'_0$	+10	+10	+22	+64	+42	+52	+50	+50	and so on to	+50	+50	+50	+52	+30	− 2	$+10\sum_{x=w+1}^{x=w+4} l_x$
$50\sum_{x=0}^{x=w} l_x$	+50	+50	+50	+50	+50	+50	+50	+50		+50	+50	+50	+50	+50	+50	
$50\sum_{x=0}^{x=w} l_x-10N'_0$	+40	+40	+28= (30−2)	−14= (−10−4)	+8= (+10−2)	− 2	0	0	0	0	0	0	− 2	+20	+52	$-10\sum_{x=w+1}^{x=w+4} l_x$

or $40(l_0+l_1)+30l_2-10l_7+10l_{12}+20l_w-2(l_2+2l_7+l_{12}+l_{17}+l_{w-5})-10\sum_{x=w+1}^{x=w+4} l_x$, when $l_{w+5}=0$.

SUM OF SURVIVORS IN FIVE-YEAR GROUPS AND AT EACH FIFTH YEAR OF AGE AND OVER.

CALCULATION OF LIFE TABLE FOR MALES IN THE STATE OF NEW YORK: 1909–1911.

PHOTOGRAPHS OF ADDING MACHINE TAPES UPON WHICH CALCULATIONS WERE MADE.

40 *Continued*

Computations of $N'_{x:\overline{5}|}$

```
                                 #
   76641        784.05
   74416        717.66
 1510570
99849829
 1511456
  766410
          22
 3789322       1501.71#
                      #
                      #
   74416        766.41
   71766        684.08
 1461820
99854951
 1462953
  744160
          27
 3670066       1450.49#
                      #
                      #
   71766        74416
   68408        64396
 1401740
99861188
 1403102
  717660
          32
 3523864       138812#
                      #
                      #
   68408        717.66
   64396        59824
 1328040
99868410
 1329254
  684080
          37
 3342588       1315.90#
                      #
                      #
   64396        684.08
   59824        545.72
 1242200
99877020
 1243440
  643960
          42
 3130840       1229.80#
                      #
                      #
   59824        643.96
   54572        483.70
 1143960
99887234
 1145590
  598240
          47
 2889420       1127.66#
                      #
                      #
   545.72       59824
   48370        40815
 1029420
99899361
 1031723
  545720
          52
 2609166       100639#
                      #
                      #
   48370        545.72
   40815        322.8.8
  891850
99913140
  894175
  483700
          57
 2272050        868.60#
                      #
```

```
                                  #
    40815          483.70
    32288          233.67
   731030
 99928263
   732396
   408150
             62
  1872942          717.37#
                         #
                         #
    32288          40815
    23367          14595
   556550
 99944590
   556795
   322880
             67
  1436470          55410#
                        #
                        #
    23367          322.88
    14595           7317
   379620
 99960395
   377977
   233670
             72
   989624          396.05#
                         #
                         *
    14595          233.67
     7317           26.74
   219120
 99973959
   214991
   145950
             77
   575932          260.41#
                         #
                         #
     7317          14595
     2674            5.76
    99910
 99984829
    94730
    73170
             82
   262630          151.71#
                         #
                         #
     2674           7317
      576            .48
    32500
 99992635
    28385
    26740
             87
    83510           73.65#
                         #
                         #
      576           26.74
       48              1
     6240
 99997325
     4189
     5760
             92
    14138           26.75*
                         #
                         #
       48            5.76
        1
      490
 99999424
 99999963
      480
             97
      406            5.76#
```

41

x	l_x	N'_x
102	1	# 10 4.06
97	48	416s 14138
92	576	1455.4s 83510
87	2674	980.64s 2626.30
82	7317	360694s 5759.32
77	14595	936626s 989624
72	23367	1926250s 14364.70
67	32288	3362720s 18729.42
62	40815	5235662s 2272050
57	48370	7507712s 26091.66
52	54572	101168.78s 2889420
47	59824	13 6298s 31308.40
42	64396	16137138s 33425.88
37	68408	19479726s 35238.64
32	71766	23 35.90s 36700.66
27	74416	26673656s 3789322
22	76641	30462978s 3887210
17	78405	3435.01.88s 3954690
12	79510	383048.78s 4 4012
7	80629	42308890s 4125408
2	84263	46434298s 872510
1	87251	47306808s 1
0	100000	48306808#

COMPLETE EXPECTATION OF LIFE.

CALCULATION OF LIFE TABLE FOR MALES IN THE STATE OF NEW YORK: 1909-1911.

PHOTOGRAPHS OF ADDING MACHINE TAPES UPON WHICH CALCULATIONS WERE MADE:

42

Check on tapes 39 to 41 *

1872510	
1872510	
1872510	
1872510	
842630	
842630	
842630	842.63
	806.29
99193710	806.29
795100	795.10
10	784.05
10	48
	4034.84
99193032	
9199792	8069.68 S
48306808	
(8) 57506600	8069.68 *
	*
	*
	115013.20
	115013.20
	115013.20
	115013.20
	115013.20
	(8) 57,5066.00 *

43

x	N'_x/l_x	$\mathring{e}_x$ *
0	4831	47.81
1	5422	53.72
2	5511	54.61
7	5247	51.97
12	4818	47.68
17	4381	43.31
22	3975	39.25
27	3584	35.34
32	3205	31.55
37	2848	27.98
42	2506	24.56
47	2174	21.24
52	1854	18.04
57	1552	15.02
62	1283	12.33
67	1041	9.91
72	824	7.74
77	642	5.92
82	493	4.43
87	367	3.17
92	253	2.03
97	87	.37

Accordingly the rule for checking the work in tapes 39 to 41 is as follows: Split machine between banks 9 and 10.

I.—(1) Set up l_0+l_1, that is, $100{,}000+l_1$, in ten's place on the left of the machine and repeat four times.

(2) Set up $10l_2$ on the left and repeat three times, adding it in unit's place on the right with the third repetition.

(3) Set up l_7 on the right of the machine, repeating twice, and with the second repetition setting up the complement of l_7 in ten's place on the left.

(4) Set up $10l_{12}$ on the left of the machine and l_{12} on the right.

(5) Set up $10l_w$ on the left of the machine and l_{17} on the right.

(6) Set up $10l_w$ on the left of the machine and l_{w-5} on the right.

(7) Repeat total seen at right through glass at base of machine.

(8) Set up on left of machine the complement of total now seen at right through the glass at its base.

(9) Set up complement of $10\sum_{x=w+1}^{x=w+4} l_x$ on right of machine. $\sum_{x=w+1}^{x=w+4} l_x$ is zero in this table. See end of section 40.

(10) Add total of tape 41, and take a total.

II.—Repeat total of tape 39 five times in ten's place and take a total. As indicated by the marks (8) to the left of each of the totals of I and II, they should agree.

The operator, to preserve the first part of his check should his totals not agree, takes a subtotal between steps (8) and (9).

DETERMINATION OF $\mathring{e}_x$.

43. The work on this tape is generally performed in pencil on the left margin of tape 41, since the l_x are not copied there in actual practice. By putting in the ages in the right margin of tapes 39 and 41, the operator can readily find the dividend in tape 41 and the corresponding divisor in tape 39, and he can enter his quotient from the computing machine to the left of the dividend in tape 43.

When the finished tapes were no longer needed for further computations, they were pasted on a large sheet of heavy manila paper and enough headings inserted to make easy any possible future reference to them. In this way all the computations for each life table were kept in order. This paper was also easy to file away.

No knowledge of algebraic processes is needed to compute life table by the methods described in sections 16 to 45. Under proper supervision any good adding machine operator can readily learn these steps and then do all the work of computing life tables.